DERIVATION OF THE VARIANCE OF $\int_0^T W_t dt$

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Question: Is $\int_0^T W_t dt$ a normal random variable?

Approach One: Let $t_0 = 0, \cdots, t_i = i\Delta t, \cdots, t_N = N\Delta t$ where $\Delta t = \frac{T}{N}$. Define (Left point integration)

$$\begin{aligned} B_N(\Delta t) &= \sum_{i=0}^{N-1} W_{t_i}\Delta t \\ &= \sum_{i=0}^{N-1}\left(\sum_{j=0}^{i-1} W_{t_{j+1}} - W_{t_j}\right)\Delta t \\ &= \sum_{i=0}^{N-1}\sum_{j=0}^{i-1}\Delta W_{t_j}\Delta t \\ &= \Delta W_{t_0}\frac{(N-1)T}{N} + \cdots \Delta W_{t_i}\frac{(N-i-1)T}{N} + \Delta W_{t_{N-2}}\frac{T}{N} \end{aligned}$$

since every $\Delta W_{t_i} \sim N\left(0, \frac{T}{N}\right)$ and is independent of each other, we can get

$$B_N(\Delta t) \sim N\left(0, \sum_{i=0}^{N-2}(N-i-1)^2\frac{T^3}{N^3}\right) \sim N\left(0, \frac{(N-1)(2N-1)}{6N^2}T^3\right)$$

when $N \to \infty$, we can deduce that

$$\lim_{N\to\infty} B_N(\Delta t) \sim N\left(0, \frac{T^3}{3}\right)$$

Approach Two: Use direct calculation for the variance deduction

$$\begin{aligned} \mathbb{E}\left[\left(\int_0^T W_t dt\right)^2\right] &= \mathbb{E}\left[\int_0^T W_u du \cdot \int_0^T W_v dv\right] \\ &= \mathbb{E}\left[\int_0^T\int_0^T (W_u \cdot W_v)\, dudv\right] \end{aligned}$$

Keep in mind that

$$\mathbb{E}\left[\int_0^T\int_0^T (W_u \cdot W_v)\, dudv\right] = \int_0^T\int_0^T \mathbb{E}[W_u \cdot W_v]\, dudv$$

while

$$\mathbb{E}[W_u \cdot W_v] = \min\{u, v\}$$

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therefore

$$\begin{aligned}
\mathbb{E}\left[\int_0^T \int_0^T (W_u \cdot W_v)\, dudv\right] &= \int_0^T \int_0^T \min\{u,v\}\, dudv \\
&= \int_0^T \left(\int_0^v \min\{u,v\}\, du + \int_v^T \min\{u,v\}\, du\right) dv \\
&= \int_0^T \left(\int_0^v udu + \int_v^T vdu\right) dv \\
&= \int_0^T \left(\frac{v^2}{2} + v(T-v)\right) dv \\
&= \frac{T^3}{2} - \frac{T^3}{6} \\
&= \frac{T^3}{3}
\end{aligned}$$

Approach Three: Use integration by part formula

$$\int_0^T W_t dt = TW_T - \int_0^T tdW_t$$

Similarly, define

$$M_N(\Delta t) = \sum_{i=0}^{N-1} t_i \left(W_{t_{i+1}} - W_{t_i}\right)$$

It is obvious

$$M_N(\Delta t) \sim N\left(0, \sum_{i=0}^{N-1} i^2 \frac{T^2}{N^2}\frac{T}{N}\right) \sim N\left(0, \frac{(N-1)(2N-1)}{6N^2}T^3\right)$$

Therefore

$$\lim_{N\to\infty} M_N(\Delta t) \sim N\left(0, \frac{T^3}{3}\right)$$

Since

$$TW_T \sim N\left(0, T^3\right)$$

but it is dependent of $\int_0^T tdW_t$, so the distribution is not clear.

The sum of two independent normally distributed random variables is normal, with its mean being the sum of the two means, and its variance being the sum of the two variances. In the event that the variables X and Y are *jointly normally distributed* random variables, then $X+Y$ is still normally distributed (see Multivariate normal distribution) and the mean is the sum of the means. However, the variances are not additive due to the correlation. Indeed,

$$\sigma_{X+Y} = \sqrt{\sigma_X^2 + \sigma_Y^2 + 2\rho\sigma_X\sigma_Y}$$

If we assume TW_T and $\int_0^T tdW_t$ and jointly normal distributed, then we can deduce that the correlation of these two normal random variables is

$$\rho = -\frac{\sqrt{2}}{3}$$

Homework: Given two independent Brownian motion W_t and Z_t, what is the distribution of $\frac{W_t}{Z_t}$? (Hint: use Moment Generating Function to see the outcome is the Pareto distribution.)